On the Almost Sure Convergence of Syracuse Sequences

Alain Slakmon[1], Luc Macot[2]
Département de mathématiques [1], Département de physique[2]
Collège de Bois-de-Boulogne
10555 avenue de Bois-de-Boulogne
Montréal (Québec), Canada H4N 1L4
e-mail : alain.slakmon@bdeb.qc.ca
luc.macot@bdeb.qc.ca

Abstract. This paper applies theory and methods of random walks and gambling systems to a probabilistic model in which Syracuse sequences are shown to converge almost surely.

Key words: Syracuse conjecture**,** 3x+1 problem, random walk, gambling theory, number theory

Alain Slakmon
Département de mathématiques
Collège de Bois-de-Boulogne
10555 avenue de Bois-de-Boulogne
Montréal (Québec), Canada H4N 1L4

Tel. : (514) 332-3000, ext. 6610
e-mail : alain.slakmon@bdeb.qc.ca

Introduction

The *Syracuse conjecture*, also known as the *3x+1 problem*, has been extensively studied since it's formulation in the 1950's and is considered to date as still open (Delahaye 1998). Basically, the problem states that starting from S_0, a positive integer greater than one, a deterministic but essentially unpredictable finite or infinite sequence of integers $S_0, S_1, S_2, ..., S_i, ...$ is generated by applying the following rule: for any $i \in \{0,1,2,3,...\}$, if S_i is even then $S_{i+1} = \frac{1}{2} S_i$, and if S_i is odd then $S_{i+1} = 3S_i + 1$. The process stops if S_i=1 for some *i*, and the sequence is then finite. The Syracuse conjecture states that for any starting point S_0, there exists n such that S_n= 1; equivalently, all Syracuse sequences are finite. This has been shown to be true for numbers up to 2.7×10^{16} (Oliveira 1999) by direct calculations and certain classes of integers have also been shown to comply to it (Delahaye 1998). Most methods used to date in trying to prove this conjecture are based on number theory and seem to lead to roadblocks due to the complexity of that model (Lagarias 1996). However, probabilistic models have also been used, some with significant results. For an overview, see (Chamberland 2003). Our approach is entirely probabilistic. It allows us to show that, within the constraints of a formally defined probabilistic model, the conjecture is true almost surely (a.s.), that is with probability one.

Probabilistic model

This paper considers the Syracuse sequence as a random walk (Feller 1968) on the integers. It is closely related to the classical gambler's ruin problem (Feller 1968) but with a different set of parameters. The sequence $\{ S_n \}$ will be seen as the sequence of fortunes of a gambler, called the *Syracuse gambler*, or simply gambler *S*. Starting with S_0 units of wealth, gambler *S* risks at each bet half of what he owns. At bet *i* his fortune is S_i. If S_i is even, he loses and his fortune becomes $S_{i+1} = \frac{1}{2} S_i$, while if S_i is odd, he wins and his fortune becomes $S_{i+1} = 3S_i + 1$. Considering that $3S_i + 1$ is always even when S_i is odd, if gambler *S* wins one bet he will surely lose the next. Thus S_i will always lead to $\frac{3S_i + 1}{2} = \frac{3}{2} S_i + \frac{1}{2}$ after a win and the game continues from there. To keep our model simple, we define $\{ S_n \}$ recursively as:

Definition: $$S_{i+1} = \begin{cases} \frac{3}{2} S_i + \frac{1}{2}, & \text{if } S_i \text{ is odd} \\ \frac{1}{2} S_i, & \text{if } S_i \text{ is even} \end{cases}$$

Gambler *S* is considered ruined and must stop playing if his fortune drops to one unit, otherwise he must bet at every play, hoping for an unbounded increase in wealth.

Considering the long run unpredictability of the sequence of fortunes and the fact that a randomly chosen integer has equal probability of being odd or even, our **main assumption** here is that, **at each play, the Syracuse gambler has equal chances of winning or losing a bet**, disregarding any information provided by the known present position. Given that bets must be placed at every play, this information is considered as null, save for short term prediction. In this model, gambler S is in a favorable game situation since a win brings in an extra $\frac{1}{2}$ unit, giving a positive expected win of $\frac{1}{4}$ units at each play. It can be thought that this advantage, however small, could possibly lead gambler S to riches almost surely or at least with a positive probability, but this is not the case. We show here that the random sequence $\{S_n\}$ leads almost surely to one, and does so in a finite number of steps.

In order to prove this, we rely on the help of a second gambler (T), playing a similar game, synchronized with gambler S as for winning or losing a bet, but with some game parameters set differently. These will allow for gambler T, starting from the same point $T_0 = S_0$, to always be at a level of wealth equal or above that of gambler S. Thus, if $\{T_n\}$ represents the random sequence of fortunes of gambler T, we will arrange to have $T_i \geq S_i$ for all i. Moreover, using Theorem 1 (below), we show that for gambler T, with suitably chosen game parameters, $T_n \to 0$ a.s. . This implies the ruin gambler T and that of gambler S with a probability of one.

To construct the sequence $\{T_n\}$, we first define a sequence of random variables $X_0, X_1, ..., X_i, ..,$ independent and identically distributed (i.i.d.) as:

Definition $X = \begin{cases} \gamma > 1 & \text{with probability } p = \frac{1}{2} \\ -1 & \text{with probability } q = \frac{1}{2} \end{cases}$

Note that here $E(X) = p\gamma - q = \frac{1}{2}(\gamma - 1) > 0$, a favorable game to play.

Now, For any $\sigma \in (0,1)$, define T_i ($i = 0,1,2,3,...$) such that:

Definition $T_{i+1} = \begin{cases} T_i(1+\sigma\gamma) & \text{if } X_i = \gamma \\ T_i(1-\sigma) & \text{if } X_i = -1 \end{cases}$

The sequence $\{T_n\}$ can be considered as the sequence of fortunes of gambler *T*, playing according to the **strategy** σ, whereby at each play he bets a fixed constant percentage (σ) of his present wealth T_i. If he wins, his fortune increases to $T_{i+1} = T_i(1+\sigma\gamma)$, otherwise it is reduced to $T_{i+1} = T_i(1-\sigma)$. Note that this gives $T_{i+1} = T_i(1+\sigma X_i)$, and by induction $T_{n+1} = T_0 \prod_{i=0}^{n}(1+\sigma X_i)$.

We aim to show that, with $\sigma = \frac{1}{2}$ and γ suitably chosen, we will have $T_n \to 0$ a.s.. To achieve this, we use the following :

Theorem 1

Let *X* and all other symbols be as previously defined.

If $E(X) = p\gamma - q > 0$ (a favorable game), there exists a value $\xi \in (0,1)$ that divides constant strategies in two classes such that, as $n \to \infty$,

1) If $\sigma \in (0,\xi)$ then $T_n \to \infty$ a.s.

2) If $\sigma \in (\xi,1)$ then $T_n \to 0$ a.s.

Note: This is equivalently stated in theorem 1 (Dubins and Savage 1960), and detailed in Section E, pp. 26-28, in (Slakmon 1979); for immediate reference, essentials of the proof are provided in Appendix I.

In our model, solving for ξ reduces to solving the equation $E(\log(1+\xi X))=0$ (see proof of theorem 1, Appendix I). Given that here $E(\log(1+\xi X)) = p\log(1+\xi\gamma)+q\log(1-\xi)$ and $p=q=\frac{1}{2}$, we obtain $\xi = 1-\frac{1}{\gamma}$. Then, considering that $\sigma=\frac{1}{2}$, setting γ such that $1-\frac{1}{\gamma}<\frac{1}{2}$ will imply $T_n \to 0$ a.s. .

Main result

We will now show that in this model it is possible to set γ such that both this condition and $T_i \geq S_i$ for all i, can be met. This will imply the ruin of gambler T and, by consequence, that of gambler S as well. We propose:

Theorem 2

Syracuse sequences converge almost surely

Proof

Let the sequence $\{T_n\}$ be as previously defined. Let $B>1$ be an integer lower bound such that any integer value smaller or equal to B is known to validate the conjecture. B will be considered as a sufficient lower bound for proof and can be set as small as 2 or as high as 2.7×10^{16}, or up to whatever the present upper bound of proven values is. Gambler T is considered ruined and stops betting if his fortune at any time reaches or falls below B. Thus we have $T_i \geq B$, for any i, until the game stops.

- To achieve $T_i \geq S_i$ for all i, we need to set γ such that: $\frac{T_i}{2}\gamma \geq \frac{T_i}{2}+\frac{1}{2}$ for all i.

This reduces to $\gamma \geq 1+\frac{1}{T_i}$. It can be obtained, for all i values, by setting $\gamma \geq 1+\frac{1}{B}$.

Then, if $T_i \geq S_i$ we have:

$T_{i+1} = T_i + \frac{T_i}{2}\gamma \geq T_i + \frac{T_i}{2}\left(1+\frac{1}{B}\right) = T_i + \frac{T_i}{2} + \frac{T_i}{2B} \geq T_i + \frac{T_i}{2} + \frac{1}{2} \geq S_i + \frac{S_i}{2} + \frac{1}{2} = S_{i+1}$, after a win, and

$T_{i+1} = \frac{T_i}{2} \geq \frac{S_i}{2} = S_{i+1}$ after a loss.

- To have $T_n \to 0$ a.s. , as per Theorem 1, we also need to have γ such that: $\sigma = \frac{1}{2} > \xi = 1-\frac{1}{\gamma}$.

If $\gamma \leq 1+\frac{1}{B}$ then $\xi = 1-\frac{1}{\gamma} \leq 1-\frac{1}{1+\frac{1}{B}} = \frac{1}{B+1} < \frac{1}{2}$, for any $B > 1$.

Both conditions are thus met by setting $\gamma = 1+\frac{1}{B}$, for any valid lower bound $B > 1$.

Suppose now that gambler T plays along with gambler S, starting from a same position, $T_0 = S_0$, any integer greater that B. Both players win or lose at the same time, except that gambler T will always be as wealthy or wealthier than gambler S, with $T_i \geq S_i$, at any moment of the game. But, for gambler T, the strategy σ played is a losing one since $\sigma = \frac{1}{2} > \xi$ implies $T_n \to 0$ almost surely, as per Theorem 1. The fact that gambler T stops playing if his fortune reaches or falls below B, and the fact that $\sigma = \frac{1}{2}$ for all i, implicitly impose a strict lower bound of $\frac{B}{2}$ on bets for the duration of the game. This insures that the game will stop after a finite number of steps since the sequence $\sigma T_n \to 0$ as $n \to \infty$, the successive bets will, for some n, reach or fall below that bound. At that point, the Syracuse gambler will also have been ruined if we consider B as a valid lower bound for that player also. It follows that, in this probabilistic model of the Syracuse problem, convergence to one in a finite number of steps is almost sure. □

Discussion

Our main result is in line with those of Terras 1978, Crandall 1978, Everett 1977, and Lagarias 1996 although our approach to the problem is notably different. Moreover, our result clearly validates almost sure convergence of Syracuse sequences, while those of Terras 1978 and Everett 1977 show the existence of a limiting density of one for the set of integers which eventually lead to a value lower than the starting point. Crandall does present a heuristic argument that *lends credibility to the main conjecture*, using a smoothed stopping time function but in his own words it lacks precision. Another notable difference is that we consider sequences to converge when they reach a predetermined proven lower bound B, while other probabilistic arguments consider as convergent a sequence reaching a value smaller than the starting point.

The speed at which typical $\{T_n\}$ sequences converge depends on the value of ξ, more specifically its distance from the constant σ. Increasing the value of the lower bound B decreases ξ and accelerates the downward movement. In our model, setting $B = 2$ gives $\xi = \frac{1}{3}$, while $B = 17$ gives $\xi = \frac{1}{18}$, as compared to $\sigma = \frac{1}{2}$. This shows that the downward trend is asymptotically accelerated, which can easily be verified experimentally by computer simulation. It is worth noting that even if we increase the chances of winning each play (p), up to a certain upper bound, gambler T would still be eventually ruined. Increasing the value of p means changing our main assumption, that of a uniform distribution (mod 2) of the integers encountered in a sequence $\{S_n\}$. The existence of non uniform sequences is an underlying question in the original Syracuse problem. Increasing p increases the value of ξ, but it suffices to set B to a valid higher value to offset this advantage. For example, with $p = \frac{3}{5}$, $q = \frac{2}{5}$ and $B = 17$, solving numerically for ξ gives $\xi \cong 0.4310421933 < \sigma = \frac{1}{2}$, and σ remains a losing strategy. The strict upper bound value for p implying $\xi < \sigma = \frac{1}{2}$ is $\log_3 2 \cong 0.6309297534$. It is obtained by solving for p the equation $E(\log(1+\sigma X)) = p\log(1+\sigma\gamma) + q\log(1-\sigma) = 0$, with $\sigma = \frac{1}{2}$, $\gamma = 1$ and $q = 1 - p$. This bound was obtained by another approach in “On the 3x+1 Problem”, Crandall 1978; it also coincides with another bound derived from the “Divergent Trajectories Conjecture”, Lagarias 1996.

Our model is general enough to allow the investigation of generalized versions of the Syracuse problem. For example, Crandall cites the unsolved case of the 7x+1 problem for which it is conjectured that the associated sequences diverge. It is easy to see that if one sets $\gamma = 5$ in our model (this could be smaller and it neglects the "+1"), the corresponding T_n sequence which is always below the associated "Syracuse 7x+1" sequence, diverges almost surely. In this case, the ξ value is $1 - \frac{1}{\gamma} = \frac{4}{5}$ well over $\sigma = \frac{1}{2}$, implying divergence of "7x+1" sequences, as per theorem 1.

Conclusion

We have considered here a probabilistic analog of the Syracuse problem and showed that, in the proposed model, the conjecture is true, almost surely that is. Our proof is based on existing results of probability theory, more specifically that of random walks and gambling theory. This allowed us to use a model where the long term unpredictable sequence of values generated in the Syracuse problem was first considered as a random walk on the positive integers with a specific set of parameters and rules. Our main assumption was to suppose that, from any position, the next move could be in either direction with equal probability. This approach disregards locally available information that could possibly be applied to short term prediction of the behavior of the sequence. But, by keeping our model relatively simple, important insight is gained on the asymptotic behavior of Syracuse sequences. Although this result does not prove the non existence of exceptional sequences increasing without bounds, it shows how Syracuse sequences behave *en masse*, pulled in a downward trend.

The small advantage given to the Syracuse gambler, that extra half a unit per win, is not enough to allow for an unbounded increase in wealth. The problem is that too large a percentage of the available capital is placed on each bet, a losing strategy in this context. With the help of a second gambler, introduced as a technique to allow the use of existing results, we showed that the Syracuse gambler will eventually be ruined, that is *back to square one with probability one*. The question of whether there exists a winning strategy for the Syracuse gambler has not been looked into as it leads to a random walk on rational or real numbers, away from the integer context of the original problem. However, we have shown that the technique used here can be easily adapted and applied to similar types of problems, or generalizations of the same. The gambling language used, being picturesque, facilitates the communication and understanding of otherwise complex concepts.

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Appendix I

Theorem 1

Let X and all other symbols be as previously defined.

If $E(X) = p\gamma - q > 0$ (a favorable game), there exists a value $\xi \in (0,1)$ that divides constant strategies in two classes such that, as $n \to \infty$,

1) If $\sigma \in (0,\xi)$ then $T_n \to \infty$ a.s.

2) If $\sigma \in (\xi,1)$ then $T_n \to 0$ a.s.

Proof

The growth rate of the sequence of T_i's is given here by $E(\log T_n) = \log T_0 + \sum_{i=0}^{n-1} E(\log(1+\sigma X_i))$ (see (Breiman 1961)). Given that the sequence $\{X_i\}$ is i.i.d., analysis of the function $E(\log T_n)$ reduces to looking at the behavior of $E(\log(1+\sigma X)) = p\log(1+\sigma\gamma) + q\log(1-\sigma)$. Using $p\gamma - q > 0$ and basic calculus, it is straightforward to show that, with respect to σ, this function is positive and strictly increasing on the interval $(0, p-\frac{q}{\gamma}]$, reaching a maximum at $p-\frac{q}{\gamma}$, and then strictly decreasing on $\left[p-\frac{q}{\gamma},1\right)$, going to $-\infty$ as $\sigma \to 1$. For insight, the value $\sigma^* = p-\frac{q}{\gamma} \in (0,1)$ is considered as an essentially unique optimal strategy ((Breiman 1961), (Slakmon 1979)), that which produces $T_n \to \infty$ a.s. with the fastest rate of growth. By continuity of $E(\log(1+\sigma X))$, there is a value $\xi \in \left(p-\frac{q}{\gamma},1\right)$ such that $\mathrm{E}(\log(1+\xi X)) = 0$. Moreover, if $\sigma \in (0,\xi)$ we have $\mathrm{E}(\log(1+\sigma X)) > 0$, and if $\sigma \in (\xi,1)$ then $\mathrm{E}(\log(1+\sigma X)) < 0$.

Consider the case $\sigma \in (0,\xi)$: The random variables $\log(1+\sigma X_i)$ being i.i.d., by the strong law of large numbers, we have almost surely $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log(1+\sigma X_i) = E(\log(1+\sigma X)) > 0$, implying $\sum_{i=0}^{n-1} \log(1+\sigma X_i) \to \infty$ a.s., which in turn implies $\log T_n \to \infty$ a.s., and then $T_n \to \infty$ a.s..

The case $\sigma \in (\xi, 1)$ is treated similarly. We have:

a.s. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(1+\sigma X_i) = E(\log(1+\sigma X)) < 0$, implying $\log T_n \to -\infty$ a.s. and in turn $T_n \to 0$ a.s.. □

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